Interest rates & APR

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1 Introduction

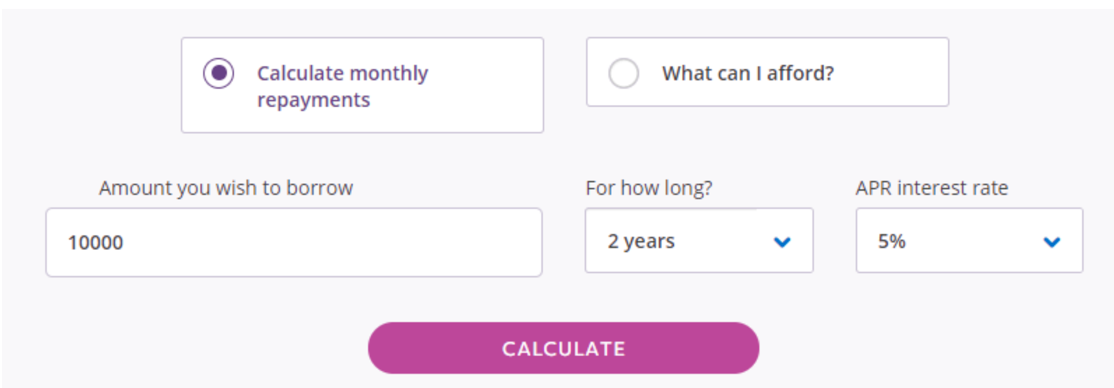
This is a short document that attempts to explain the differences between results from a number of online loan calculators, and goes a little way to explaining the maths behind those differences.

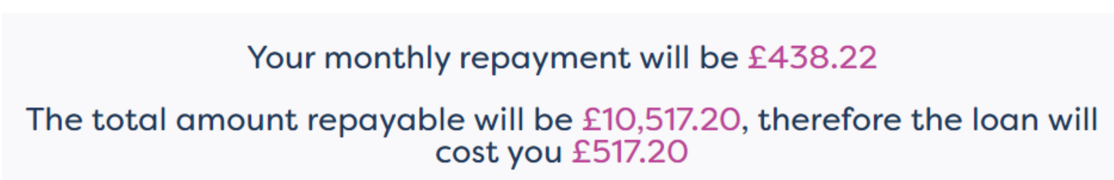
> Quite simply, we want to know: If we borrow **£10,000** for **24** months, at an interest rate of **5%** per year, what will our monthly repayment be?

We put 'loan repayment calculator' into Google search, and filled out the above details on four websites.

1.1 Moneysupermarket

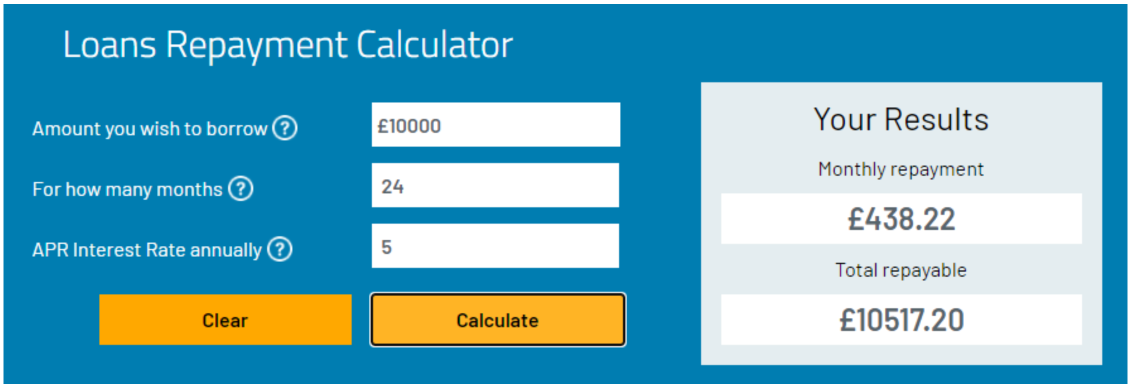
https://www.moneysupermarket.com/loans/calculator/





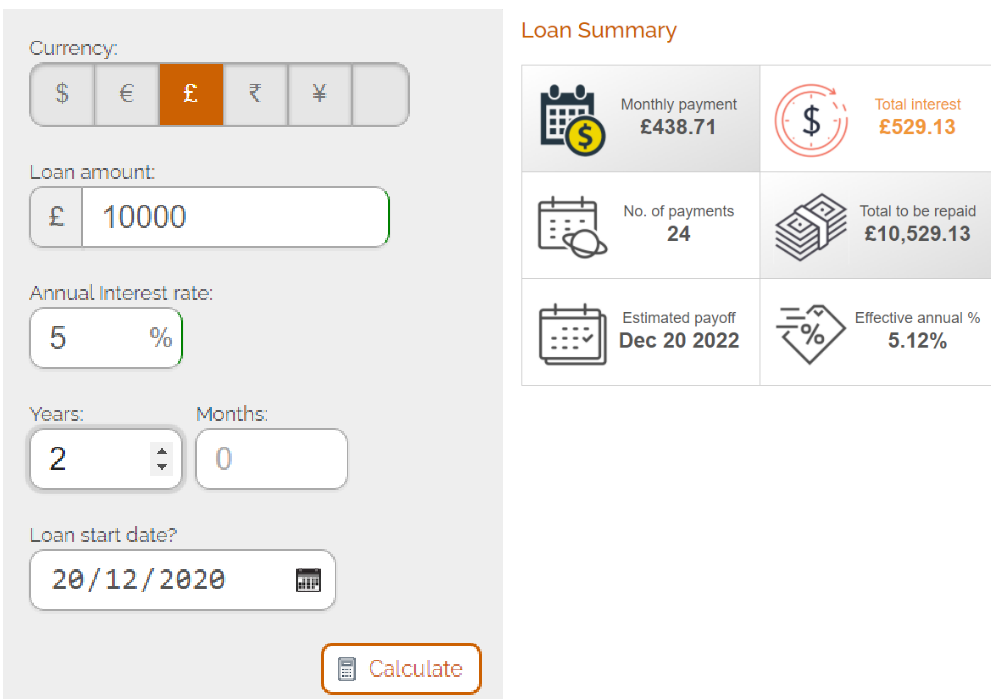
1.2 Moneyfacts

https://moneyfacts.co.uk/loans/loan-calculator/



1.3 The Calculator Site

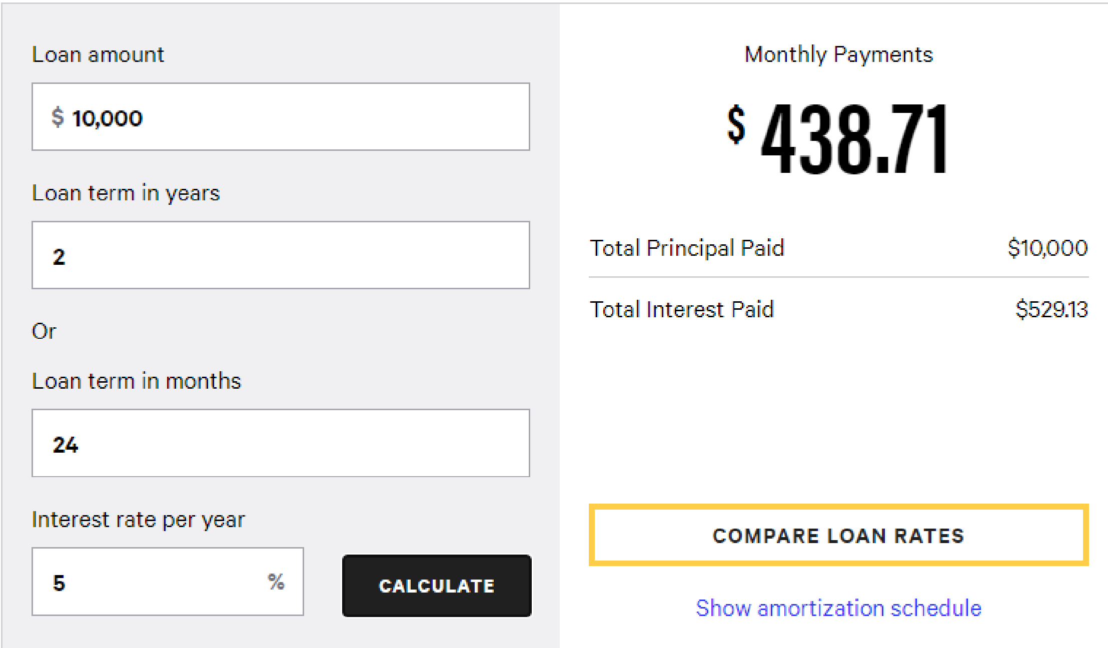
https://www.thecalculatorsite.com/finance/calculators/loancalculator.php



Note that the ‘Loan start date’ is irrelevant for our purposes.

1.4 Bankrate

https://www.bankrate.com/calculators/mortgages/loan-calculator.aspx



1.5 Comparison

The first two sites (both UK), give a monthly repayment amount of **£438.22**. The latter two sites (both US) give a figure of **£438.71**. If you look closely, both UK sites mention 'APR Interest Rate', whereas the US sites have 'Annual Interest Rate' or 'Interest rate per year'. This is the reason for the difference between the two figures, but what does it mean?

1.6 Amortised loans

All the loans referred to in this document are **amortised loans**. From investopedia:

> An amortized loan is a type of loan that requires the borrower to make scheduled, periodic payments that are applied to both the principal and interest. An amortized loan payment first pays off the interest expense for the period; any remaining amount is put towards reducing the principal amount.

The **loan schedule**, *aka* **amortisation schedule** is:

> ... a table that shows each periodic loan payment that is owed, typically monthly, and how much of the payment is designated for the interest versus the principal.

(again from investopedia) We will see some examples of these loan schedules later on.

1.6.1 APR

APR stands for 'Annual Percentage Rate'. From https://www.cashfloat.co.uk/apr-explained/

> APR is a comparative measure to help understand different loans. APR is the interest rate in addition to fees and charges over a whole year as opposed to monthly interest rates.
>
> There is a big difference between APR and interest rates. The APR includes additional fees that you might be charged on top of the interest rate. If your unsecured short term loan UK lender agreed any additional fees with you, these will be included in the APR. The APR represents the total cost of the loan to you, explained on a per year basis.

But none of the examples we gave included additional fees. So why are the figures different? This is because the formula used to calculate APR varies between countries.

1.6.2 Representative APR

There is another APR figure shown by lenders, the 'Representative APR'. This is not another different calculation but rather:

> Where credit cards or loans use a representative APR, this means 51% of successful applicants will be given the stated interest rate.
>
> If a loan is advertised as being 7.5% representative APR, this means 51% of accepted applicants have to get 7.5% as their rate. The other 49% could get a different rate (which is usually higher).

(from moneysavingexpert.com/loans/personal-loans-apr-examples/)

The 'Representative APR' value is irrelevant from here onwards.

2 Worked examples and equations

2.1 Monthly interest rate: worked example

For our given inputs (loan amount of £10,000 taken out over 24 months, at an interest rate of 5% per year, which we take to be $5\%/12 = 0.41\overline{6}\%$ per month), we'll work through the figures and derive the equations.

Firstly, here's the full loan schedule, where the interest is simply calculated on the outstanding principal balance:

£10000 at 5% per year over 24 months						
Month	Starting balance	Interest charged	Payment	Interest paid	Principal paid	End balance
1	10000.00	41.67	438.71	41.67	397.05	9602.95
2	9602.95	40.01	438.71	40.01	398.70	9204.25
3	9204.25	38.35	438.71	38.35	400.36	8803.89
4	8803.89	36.68	438.71	36.68	402.03	8401.86
5	8401.86	35.01	438.71	35.01	403.71	7998.15
6	7998.15	33.33	438.71	33.33	405.39	7592.76
7	7592.76	31.64	438.71	31.64	407.08	7185.69
8	7185.69	29.94	438.71	29.94	408.77	6776.91
9	6776.91	28.24	438.71	28.24	410.48	6366.44
10	6366.44	26.53	438.71	26.53	412.19	5954.25
11	5954.25	24.81	438.71	24.81	413.90	5540.34
12	5540.34	23.08	438.71	23.08	415.63	5124.71
13	5124.71	21.35	438.71	21.35	417.36	4707.35
14	4707.35	19.61	438.71	19.61	419.10	4288.25
15	4288.25	17.87	438.71	17.87	420.85	3867.41
16	3867.41	16.11	438.71	16.11	422.60	3444.81
17	3444.81	14.35	438.71	14.35	424.36	3020.45
18	3020.45	12.59	438.71	12.59	426.13	2594.32
19	2594.32	10.81	438.71	10.81	427.90	2166.41
20	2166.41	9.03	438.71	9.03	429.69	1736.73
21	1736.73	7.24	438.71	7.24	431.48	1305.25
22	1305.25	5.44	438.71	5.44	433.28	871.97
23	871.97	3.63	438.71	3.63	435.08	436.89
24	436.89	1.82	438.71	1.82	436.89	0.00
		529.13	**10529.13**	**529.13**	**10000.00**	

Note:

- The figures above are shown rounded to 2 d.p. but they are not rounded during the calculations. For a practical loan schedule, you would have to round the interest charged, which means the final payment is usually a few pence different from the standard monthly payment.
- In our simple example, the interest charged each month is immediately paid off; that is, the monthly payment is always greater than the interest charged, so we do not need to separately keep track of the interest balance. This may not be the case for loan schedules where there are initial interest/charges balances, or the periods are of unequal duration.
- There are other schemes for calculating the interest due, e.g. The Rule of 78 in which a greater proportion of the total interest is charged upfront, leading to more profit for the lender if the loan is settled early.

The figures in the previous loan schedule can be represented in the following way. Let L_n be the balance at the end of period n, and hence L_0 is the starting balance at the start of period 1. Let r be the monthly interest rate, and p be the monthly payment. Then:

$$\begin{aligned} L_1 &= L_0 + (L_0 \times r) - p \\ &= L_0 \times (1 + r) - p \\ &\dots \\ L_n &= L_{n-1} \times (1 + r) - p \end{aligned} \tag{1}$$

Solving this recurrence relation (see Appendix A) gives:

$$L_n = c(r+1)^{n-1} + \frac{(p - p(r+1)^n)}{r}$$

where c is a constant. Since we know L_0, we can work out c in the above, and substitute to give us:

$$L_n = L_0(r+1)^n + \frac{p(1 - (r+1)^n)}{r} \tag{2}$$

This lets us calculate directly the entries in the 'End balance' column in our previous table. For example, with $L_0 = 10000, p = 438.71, r = 0.05/12$, and $n = 13$, we have:

$$\begin{aligned} L_n &= 10000 \times (1 + 0.05/12)^{13} + \frac{438.71 \times (1 - (1 + 0.05/12)^{13})}{0.05/12} \\ &= 10555.41739 - 5848.063881 \\ &= 4707.353509 \end{aligned}$$

which is the 'End balance' for month 13.

Finally, to get an equation in terms of the payment, p, we can substitute $L_n = 0$ in equation (2) (as the principal is always paid off at the end of the loan) and solve for p:

$$p = L_0 \times \frac{r(1+r)^n}{(1+r)^n - 1} \tag{3}$$

2.1.1 Example

Using our equation (3) above, let's calculate the monthly payment and schedule for a loan amount of £6,000 at 4% per year over 8 months:

$$\begin{aligned} p &= 6000 \times \frac{(0.04/12) \times (1 + 0.04/12)^8}{(1 + 0.04/12)^8 - 1} \\ &= 761.2936767 \end{aligned}$$

The loan schedule is then:

£6000 at 4% per year over 8 months						
Month	Starting balance	Interest charged	Payment	Interest paid	Principal paid	End balance
1	6000.00	20.00	761.29	20.00	741.29	5258.71
2	5258.71	17.53	761.29	17.53	743.76	4514.94
3	4514.94	15.05	761.29	15.05	746.24	3768.70
4	3768.70	12.56	761.29	12.56	748.73	3019.97
5	3019.97	10.07	761.29	10.07	751.23	2268.74
6	2268.74	7.56	761.29	7.56	753.73	1515.01
7	1515.01	5.05	761.29	5.05	756.24	758.76
8	758.76	2.53	761.29	2.53	758.76	0.00
		90.35	**6090.35**	**90.35**	**6000.00**	

2.2 APR

The APR calculation used in the UK is based on **Net Present Value** (sometimes abbreviated to NPV) – see investopedia.com.

From the FCA Handbook MCOB 10.3 Formula and assumptions for calculating the APR:

> "The APR must be calculated so that . . . the annual percentage rate of charge is the rate for i which satisfies the equation set out in MCOB 10.3.1A R, expressed as a percentage."

where 'MCOB 10.3.1A R' is:

$$\sum_{K=1}^{K=m} \frac{A_K}{(1+i)^{t_K}} = \sum_{K'=1}^{K'=m'} \frac{A'_{K'}}{(1+i)^{t_{K'}}} \tag{4}$$

where

- K is the number identifying a particular advance of credit;
- K' is the number identifying a particular instalment;
- A_K is the amount of advance K;
- A'_K is the amount of instalment K';
- $\sum$ represents the sum of all terms indicated;
- m is the number of advances of credit;
- m' is the total number of instalments;
- t_K is the interval, expressed in years, between the *relevant date* and the date of the second advance and those of any subsequent advances numbers three to m; and
- $t_{K'}$ is the interval, expressed in years, between the *relevant date* and the dates of instalments numbered one to m'.

For our purposes, this is over-complicated. We're only considering one advance, at the start of the loan, so the left-hand side of equation (4) is just L_0, the initial loan amount. Also using the fact that we have only one payment value, we can simplify the equation to:

$$L_0 = \sum_{k=1}^{k=n} \frac{p}{(1+i)^{k/12}} \tag{5}$$

where

- L_0 is the initial loan amount
- k is an indexing variable (of the month)
- p is the monthly payment
- i is the APR value
- n is the total number of months over which the loan is paid back

This reduces to:

$$L_0 = p \times \left(\frac{(1+i)^{-n/12} - 1}{1 - (1+i)^{1/12}} \right) \tag{6}$$

Note: As mentioned earlier, the final payment is usually a few pence different due to rounding. Strictly, this should be taken into account when calculating the APR.

(Details on the US version of the APR calculation can be found at investopedia.)

2.2.1 Loan schedule and APR calculation

£10000 at 5% per year over 24 months						
Month (k)	Starting balance	Interest charged	Payment	End balance	$k/12$	$p/(1+i)^{k/12}$
1	10000.00	41.67	438.71	9602.95	1/12	436.89
2	9602.95	40.01	438.71	9204.25	2/12	435.08
3	9204.25	38.35	438.71	8803.89	3/12	433.28
4	8803.89	36.68	438.71	8401.86	4/12	431.48
5	8401.86	35.01	438.71	7998.15	5/12	429.69
6	7998.15	33.33	438.71	7592.76	6/12	427.90
7	7592.76	31.64	438.71	7185.69	7/12	426.13
8	7185.69	29.94	438.71	6776.91	8/12	424.36
9	6776.91	28.24	438.71	6366.44	9/12	422.60
10	6366.44	26.53	438.71	5954.25	10/12	420.85
11	5954.25	24.81	438.71	5540.34	11/12	419.10
12	5540.34	23.08	438.71	5124.71	12/12	417.36
13	5124.71	21.35	438.71	4707.35	13/12	415.63
14	4707.35	19.61	438.71	4288.25	14/12	413.90
15	4288.25	17.87	438.71	3867.41	15/12	412.19
16	3867.41	16.11	438.71	3444.81	16/12	410.48
17	3444.81	14.35	438.71	3020.45	17/12	408.77
18	3020.45	12.59	438.71	2594.32	18/12	407.08
19	2594.32	10.81	438.71	2166.41	19/12	405.39
20	2166.41	9.03	438.71	1736.73	20/12	403.71
21	1736.73	7.24	438.71	1305.25	21/12	402.03
22	1305.25	5.44	438.71	871.97	22/12	400.36
23	871.97	3.63	438.71	436.89	23/12	398.70
24	436.89	1.82	438.71	.00	24/12	397.05
		529.13	**10529.13**			**10000.00**

With our payment $p = 438.71$, the value for i that ensures the final column sums to £10,000 is **0.0511619**, i.e. 5.11619%. Following the FCA's rules on rounding (MCOB 10.3.4), the APR is **5.1%**.

2.2.2 Calculating the payment and APR

By re-arranging equation (6) it is straightforward to calculate the payment p given the APR and a loan amount:

$$p = L_0 / \left(\frac{(1+i)^{-n/12} - 1}{1 - (1+i)^{1/12}} \right) \tag{7}$$

For our original loan details, that gives:

$$\begin{aligned} p &= 10000/\left(\frac{(1+0.05)^{-24/12}}{1-(1+0.05)^{(1/12)}} \right) && (8) \\ &= 10000/\left(\frac{-0.09297052}{-0.004074124} \right) && (9) \\ &= \mathbf{438.22} \text{ to 2 dp} && (10) \end{aligned}$$

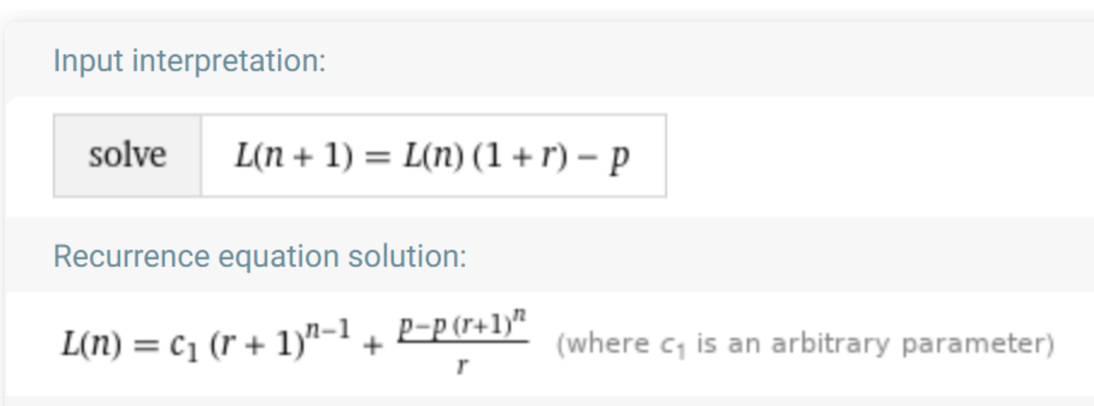
The APR i cannot be calculated directly given p and L_0. Generally, it can only be determined iteratively — see Appendix (B).

A Solving equations with Wolfram Alpha

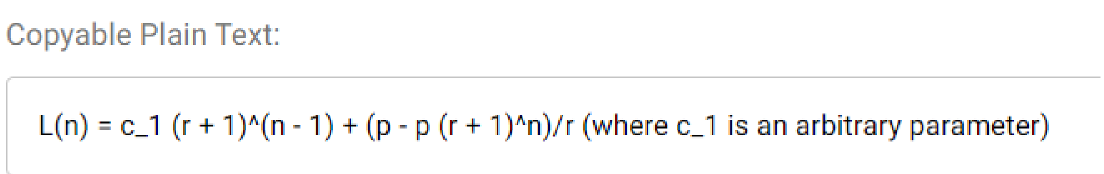
We solved equation 1 using Wolfram Alpha: https://www.wolframalpha.com/

With a slight reformatting, we paste our equation to be solved into the input box (and prefix with the word 'solve'):

AFter a few seconds, we get the result:



Hovering over the solution, we get a tab titled 'Plain Text', that gives us:



which we then just click to copy.

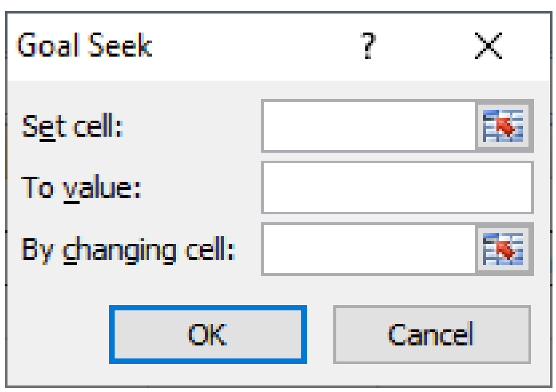
Wolfram Alpha attempts to interpret variable names in a way that makes most sense to it; e.g., i might be interpreted as $\sqrt{-1}$, and $t1, t2, t3$ might be interpreted as t, t^2, t^3. If the solution is not what you expected, swapping variable names often helps.

B Iterative solution to APR equation

We can use the `optim` function in `R` to find i:

```
L0 <- 10000;       # initial loan amount
p  <- 438.7138973; # monthly payment
n  <- 24;          # number of months

library(boot); # library that contains the optim function

target <- L0/p;

apr_fn <- function(i) {
v <- ((1+i)^(-24/12) - 1)  / (1 - (1+i)^(1/12) );
return(abs(v - target));
}

intial_guess <- 0.05; # 5%

res <- optim(intial_guess, apr_fn, method="Brent", lower=0, upper=1);
res$par;
[1] 0.0511619
```

Alternatively, we can use the 'Goal Seek' functionality within Excel — from the menu, found under `Data / What-If Analysis / Goal Seek...`:



where the first text box refers to the cell that contains the result of the calculation (as per the function `apr_fn` from the `R` code); the second box contains 0; and the third box is the initial guess for i.